

Padé-Improved Estimates of Hadronic Higgs Decay Rates

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Abstract

Asymptotic Padé-approximant methods are utilized to estimate the $\mathcal{O}(\alpha_s^5)$ contribution to the $H \rightarrow gg$ rate and the $\mathcal{O}(\alpha_s^4)$ contribution to the $H \rightarrow b\bar{b}$ rate. The former process is of particular interest because of the slow convergence evident from the three known terms of its QCD series, which begins with an $\mathcal{O}(\alpha_s^2)$ leading-order term. The $\mathcal{O}(\alpha_s^5)$ contribution to the $H \rightarrow gg$ rate is expressed as a degree-3 polynomial in $L \equiv \ln(\mu^2/m_t^2(\mu))$. We find that asymptotic Padé-approximant predictions for the coefficients of L , L^2 , and L^3 are respectively within 1%, 2%, and 7% of true values extracted via renormalization-group methods. Upon including the full set of next-order coefficients, the $H \rightarrow gg$ rate is found to be virtually scale-independent over the $0.3 M_H \lesssim \mu \lesssim M_t$ range of the renormalization scale-parameter μ . We conclude by discussing the small $\mathcal{O}(\alpha_s^4)$ contribution to the $H \rightarrow b\bar{b}$ rate, which is obtained from a prior asymptotic Padé-approximant estimate of the $\mathcal{O}(\alpha_s^4)$ contribution to the quark-antiquark scalar-current correlation function.

1. Introduction

The Higgs boson characterizes the electroweak symmetry breaking underlying the Standard Model. Its discovery and phenomenology will be of immense importance in clarifying our understanding of this symmetry breaking, as well as in providing vital information as to the nature of beyond-the-Standard-Model physics. The two leading hadronic decay modes of a Weinberg-Salam Higgs boson (H) with mass between 100 GeV and 160 GeV are the QCD processes $H \rightarrow b\bar{b}$ and $H \rightarrow$ two gluons (gg). Although the rate for this latter process is known to $\mathcal{O}(\alpha_s^4)$, such precision incorporates only two non-leading orders of a slowly converging series in the strong coupling. If $M_H = 100$ GeV, for example, the known order-by-order QCD corrections have been calculated [1] to be $1 + 0.66 + 0.21$.

The work presented here is primarily directed toward obtaining a more precise estimate of the $H \rightarrow gg$ decay rate. We utilize renormalization-group (RG) and asymptotic Padé-approximant methods to estimate the full next-order contribution to the underlying correlation function for this process. Such an approach has already been applied to the quark-antiquark scalar-current correlation function underlying the $H \rightarrow b\bar{b}$ rate [2], a calculation we review in Section 5 of the present paper.

As in our prior analysis of the two-gluon decay amplitude of a non-Standard-Model CP-odd Higgs field [3], the approach we take here is to test asymptotic Padé-approximant estimates against RG-accessible coefficients within the next-order of perturbation theory. We show such estimates to be accurate up to relative errors of only a few percent, supporting the credibility of the same approach in estimating the RG-*inaccessible* coefficient needed to determine the full $\mathcal{O}(\alpha_s^5)$ contribution to the $H \rightarrow gg$ rate.

We operate within the context of \overline{MS} expressions for the $H \rightarrow gg$ and $H \rightarrow b\bar{b}$ rates that explicitly depend on an arbitrary renormalization scale μ . It has been argued elsewhere [4, 5, 6] that asymptotic Padé-approximant methods reduce the explicit scale-dependence of perturbative quantities which must ultimately be scale invariant. We find this to be the case for the $H \rightarrow gg$ rate as well, despite residual scale-sensitivity anticipated from the estimated RG-inaccessible coefficient of the nonlogarithmic $\mathcal{O}(\alpha_s^5)$ term within the next-order correlator.

In Section 2, we demonstrate the explicit RG-invariance of the $H \rightarrow gg$ rate, as calculated in [1] in the $m_b \rightarrow 0$, $M_H^2 \ll 4M_t^2$ limit. This RG invariance enables one to calculate all the next-order coefficients c_k of $\ln^k[\mu^2/m_t(\mu^2)] (\alpha_s(\mu)/\pi)^5$ within the calculated rate; only the $k = 0$ constant term is RG-inaccessible.

In Section 3, we review how asymptotic Padé-approximant methods may be utilized to estimate this set of next-order coefficients, and demonstrate close agreement with the RG-determinations of Section 2 over the range $100 \leq M_H \leq 175$ GeV. The RG-inaccessible coefficient c_0 is also estimated over this range of Higgs masses and is fitted to its anticipated behaviour as a degree-3 polynomial in $\ln[(M_H^2/M_t^2)_{pole}]$.

In Section 4 we examine the residual scale dependence of the Padé-improved $H \rightarrow gg$ decay rate, as obtained in Section 3. Although we anticipate a relative scale dependence comparable to $c_0(\alpha_s(\mu)/\pi)^3 (\approx 3 - 10\%)$, since c_0 cannot be extracted perturbatively from the RG equation (to the order we consider), we find that the residual scale dependence of the decay rate is a full order of magnitude smaller than this estimate for μ between $0.3 M_H$ and M_t (we assume $100 \text{ GeV} \leq M_H \leq 175 \text{ GeV}$). Consequently, the scale-dependence of the Padé-derived term $c_0(\alpha_s(\mu)/\pi)^5$ very nearly cancels the scale-dependence anticipated from truncation of the perturbative series for the $H \rightarrow gg$ rate, thereby facilitating the (nearly) scale-invariant rate predictions of Table 6. Predictions of the $H \rightarrow gg$ decay rate are tabulated both for the $m_b \rightarrow 0$, $M_H^2 \ll 4M_t^2$ limit, and for the leading-order departures from this limit for appropriate constant and running values for m_b and M_t .

In the concluding section, we review how prior asymptotic Padé-approximant estimates of the $\mathcal{O}(\alpha_s^4)$ contribution to the scalar-current correlation function [2] can be incorporated into the $H \rightarrow b\bar{b}$ decay rate. We demonstrate that the resulting next-order contribution, though only 0.01% of the leading term, is somewhat *larger* than known $\mathcal{O}(\alpha_s^2)$ power-suppressed contributions [7]. Consequently, the estimated $\mathcal{O}(\alpha_s^4)$ term enables the $H \rightarrow b\bar{b}$ rate to be estimated to four significant figure accuracy.

2. RG-Invariance of the $H \rightarrow gg$ Rate

The Higgs $\rightarrow gg$ decay rate is given explicitly to 3-loop order in [1] by the following expressions:

$$\Gamma_{H \rightarrow gg} = \frac{\sqrt{2}G_F}{M_H} R(\alpha_s, q^2 = M_H^2, \mu^2, M_t^2), \quad (2.1a)$$

$$R = C_1^2 Im < [0'_1]^2 >, \quad (2.1b)$$

$$C_1 = -\frac{x^{(6)}}{12} \left[1 + x^{(6)} \left(\frac{11}{4} - \frac{1}{6} \ln \left(\frac{\mu^2}{M_t^2} \right) \right) \right. \\ \left. + (x^{(6)})^2 \left(\frac{211}{36} + \frac{55}{48} \ln \left(\frac{\mu^2}{M_t^2} \right) + \frac{1}{36} \ln^2 \left(\frac{\mu^2}{M_t^2} \right) \right) + \mathcal{O} \left[(x^{(6)})^3 \right] \right] \quad (2.1c)$$

$$Im < [0'_1]^2 > = \frac{2q^4}{\pi} \left[1 + x^{(5)} \left(\frac{149}{12} + \frac{23}{6} \ln \left(\frac{\mu^2}{q^2} \right) \right) \right. \\ \left. + (x^{(5)})^2 \left(68.64817 + \frac{1297}{16} \ln \left(\frac{\mu^2}{q^2} \right) + \frac{529}{48} \ln^2 \left(\frac{\mu^2}{q^2} \right) \right) + \mathcal{O} \left[(x^{(5)})^3 \right] \right]. \quad (2.1d)$$

In the above expression, $< [0'_1]^2 >$ is the vacuum polarization of the Higgs field induced via the gluon operator $0'_1 = G_{a\mu\nu} G_a^{\mu\nu}$ [1], and $x^{(n_f)} \equiv \alpha_s^{(n_f)}/\pi$, where $\alpha_s^{(n_f)}(\mu)$ is the running strong coupling with n_f active flavours.

Five flavours are assumed to be light in both (2.1c) and (2.1d). The t -quark mass M_t appearing in (2.1c) is an RG-invariant pole mass, and M_H^2 is assumed to be small compared to $4M_t^2$ [1].

Our goal here is to use asymptotic Padé-approximant methods in conjunction with the RG-invariance of (2.1a) in order to predict the next order contribution to $\Gamma(H \rightarrow gg)$. In a previous application [6] of such methods to the inclusive semileptonic $b \rightarrow u$ rate, it was found that success in predicting RG-accessible next-order coefficients was greatly enhanced by recasting the entire expression in terms of the running fermion mass. Indeed, such replacement of the b -quark pole mass with its scale-dependent \overline{MS} mass had already been employed by van Ritbergen [8] to avoid a renormalon pole.

Consequently, we first re-express the $H \rightarrow 2g$ rate in terms of the running t -quark mass $m_t(\mu)$, which evolves via a six-active-flavour γ_m -function, and the corresponding six-active-flavour running coupling $x^{(6)}(\mu)$. This transformation is facilitated by the following relationships [9]:

$$x^{(5)}(\mu) = x^{(6)}(\mu) - \frac{(x^{(6)}(\mu))^2}{6} \ln \left(\frac{\mu^2}{m_t^2(\mu)} \right) + \mathcal{O} \left[(x^{(6)})^3 \right], \quad (2.2)$$

$$m_t(\mu)/M_t = \left[1 - x^{(6)}(\mu) \left(\frac{4}{3} + \ln \left(\frac{\mu^2}{M_t^2} \right) \right) + \mathcal{O} \left[(x^{(6)})^2 \right] \right]. \quad (2.3)$$

Equation (2.3) implies that the logarithm in (2.1c) may be re-expressed in terms of $L \equiv \ln(\mu^2/m_t^2(\mu))$:

$$\ln \left(\frac{\mu^2}{M_t^2} \right) = L - x^{(6)}(\mu) [2L + 8/3] + \mathcal{O} \left[(x^{(6)})^2 \right] \quad (2.4)$$

With $q^2 = M_H^2$, the logarithm in (2.1d) can also be expressed entirely in terms of the running t -quark mass and a logarithm $T \equiv \ln(M_H^2/M_t^2)$ of the ratio of RG-invariant pole masses:

$$\ln \left(\frac{\mu^2}{M_H^2} \right) = L - T - x^{(6)}(\mu) [2L + 8/3] + \mathcal{O} \left[(x^{(6)})^2 \right]. \quad (2.5)$$

Substitution of (2.2) and (2.5) into (2.1d) with $q^2 = M_H^2$, and substitution of (2.4) into (2.1c) leads to the following expression for the $H \rightarrow gg$ decay rate:

$$\Gamma_{H \rightarrow gg} = \frac{\sqrt{2} G_F M_H^3}{72\pi} S \left[x^{(6)}(\mu), L(\mu), T \right], \quad (2.6)$$

$$\begin{aligned} S[x, L, T] = & x^2 \left(1 + x \left[\left(\frac{215}{12} - \frac{23T}{6} \right) + \frac{7}{2} L \right] \right. \\ & + x^2 \left[\left(146.8912 - \frac{4903}{48} T + \frac{529}{48} T^2 \right) + \left(\frac{1445}{16} - \frac{161}{8} T \right) L + \frac{147}{16} L^2 \right] \\ & \left. + x^3 [c_0 + c_1 L + c_2 L^2 + c_3 L^3] + \mathcal{O}(x^4) \right). \end{aligned} \quad (2.7)$$

In (2.7), we list the unknown coefficients c_0, c_1, c_2 and c_3 of the 4-loop contribution to the rate. Three of these may be extracted by the scale-[RG-] invariance of the physical decay rate: $d\Gamma/d\mu = 0$. This invariance implies that

$$\begin{aligned} 0 &= \mu \frac{dS}{d\mu} [x, L, T], \\ &= [1 - 2\gamma_m(x)] \frac{\partial S}{\partial L} + \beta(x) \frac{\partial S}{\partial x}. \end{aligned} \quad (2.8)$$

Both the β and γ_m functions in (2.8) are referenced to six active flavours:

$$\beta^{(6)}(x) = -\frac{7}{4}x^2 - \frac{13}{8}x^3 + \frac{65}{128}x^4 \dots \quad (2.9)$$

$$\gamma_m^{(6)}(x) = -x - \frac{27}{8}x^2 \dots \quad (2.10)$$

One can easily verify from the known terms listed in (2.7) that (2.8) is perturbatively valid to orders x^3, x^4 (including explicit cancellation of terms involving T), and $x^4 L$. The continued perturbative validity of (2.8) to orders $x^5 L^2$, $x^5 L$, and x^5 is sufficient to determine the four-loop coefficients c_1, c_2 , and c_3 :

$$c_1 = 910.3167 - \frac{16643}{24}T + \frac{3703}{48}T^2, \quad (2.11)$$

$$c_2 = \frac{1225}{4} - \frac{1127}{16}T, \quad (2.12)$$

$$c_3 = \frac{343}{16}. \quad (2.13)$$

The coefficient c_0 is not RG-accessible to these orders. In the next section, we will utilize asymptotic Padé approximant methods to estimate the four-loop coefficients $\{c_0, c_1, c_2, c_3\}$. As in prior work [2, 3, 6], the accuracy of these predictions in reproducing (2.11 - 2.13) will serve as an indication of the accuracy of our estimate for c_0 .

3. Padé-Predictions for the Four-Loop Coefficients

The asymptotic Padé-approximant procedure for estimation of the four loop coefficients c_0, c_1, c_2, c_3 in (2.7) has been delineated in previous work [2, 3, 6]. The series (2.7) may be expressed in the form

$$S[x, L, T] = x^2 [1 + R_1[L, T]x + R_2[L, T]x^2 + R_3[L, T]x^3 + \dots] \quad (3.1)$$

where R_1 and R_2 are known, and R_3 is to be determined:

$$R_1[L, T] = \left(\frac{215}{12} - \frac{23}{6}T \right) + \frac{7}{2}L \quad (3.2)$$

$$R_2[L, T] = 146.8912 - \frac{4903}{48}T + \frac{529}{48}T^2 + \left(\frac{1445}{16} - \frac{161}{8}T \right)L + \frac{147}{16}L^2 \quad (3.3)$$

$$R_3[L, T] = c_0(T) + c_1(T)L + c_2(T)L^2 + c_3L^3. \quad (3.4)$$

Initially, we shall eliminate T as a variable by assuming that the Higgs pole mass is 100 GeV, in which case $T = \ln[M_H^2/M_t^2] = 2 \ln(100/175.6) = -1.126$. As described in [6] and [10], the $[0|1]$ Padé-approximant prediction for R_2 is

$$R_2^{[0|1]} = R_1^2 \quad (3.5)$$

and the $[1|1]$ Padé-approximant prediction for R_3 is

$$R_3^{[1|1]} = R_2^2/R_1. \quad (3.6)$$

If the error of $[N|1]$ approximants in predicting R_{N+2} , the $N+2$ term in the perturbative series, is inversely proportional to $N+1$ [10] — *i.e.*, if

$$\frac{R_{N+2}^{[N|1]} - R_{N+2}^{exact}}{R_{N+2}^{exact}} = \frac{-A}{N+1}, \quad (3.7)$$

where A is a constant— one may then utilize (3.5) and the exact value for R_2 within (3.7) to obtain $A = (R_2^2 - R_1^2)/R_2$. Substituting (3.6) and this estimate for A into (3.7), one obtains an error-improved estimate for the unknown coefficient R_3 [11]:

$$R_3[L] = 2R_2^3[L]/(R_1[L]R_2[L] + R_1^3[L]) \quad (3.8)$$

To obtain estimates of the coefficients c_0, c_1, c_2, c_3 within (3.4), we match the scale dependence of (3.4) to that of (3.8) over the purely perturbative $L > 0$ region [corresponding to the ultraviolet scales $\mu > m_t(\mu)$] through use of the moment integrals

$$N_k \equiv (k+2) \int_0^1 dw w^{k+1} R_3(w), \quad (3.9)$$

where $w = m_t^2(\mu)/\mu^2$ [$L = -\ln(w)$]. Substitution of (3.4) into the integrand of (3.9) yields the following expressions for the first four moments [2]:

$$N_{-1} = c_0 + c_1 + 2c_2 + 6c_3, \quad (3.10a)$$

$$N_0 = c_0 + \frac{1}{2}c_1 + \frac{1}{2}c_2 + \frac{3}{4}c_3, \quad (3.10b)$$

$$N_1 = c_0 + \frac{1}{3}c_1 + \frac{2}{9}c_2 + \frac{2}{9}c_3, \quad (3.10c)$$

$$N_2 = c_0 + \frac{1}{4}c_1 + \frac{1}{8}c_2 + \frac{3}{32}c_3. \quad (3.10d)$$

However, explicit numerical estimates of these four moments may be obtained via substitution of (3.2) and (3.3) for the respective factors of $R_1[L]$ and $R_2[L]$ appearing in (3.8), and by subsequent substitution of this estimate for $R_3[L]$ into the integrand of (3.9) [with $L = -\ln(w)$]. For $M_H = 100$ GeV the resulting estimates are

$$N_{-1} = 5102.9, N_0 = 3542.9, N_1 = 3131.8, N_2 = 2944.7. \quad (3.11)$$

Substitution of these values into (3.10) yields the following predicted values ($c_i^{Pad\acute{e}}$) for the four-loop terms in the $H \rightarrow gg$ decay rate (2.7):

$$c_0^{Pad\acute{e}} = 2453, c_1^{Pad\acute{e}} = 1772, c_2^{Pad\acute{e}} = 378.4, c_3^{Pad\acute{e}} = 20.27. \quad (3.12)$$

The exact values of c_1, c_2 , and c_3 were determined via RG methods in the previous section. For $M_H = 100$ GeV [*i.e.* for $T = -1.126$], these values are found from (2.11-13) to be

$$c_1 = 1789.02, c_2 = 385.568, c_3 = 21.4375. \quad (3.13)$$

Comparing (3.12) and (3.13), one finds the relative error [$\delta c_i \equiv (c_i^{Pad\acute{e}} - c_i)/c_i$] of the Padé estimates for c_1, c_2 and c_3 to be -0.95%, -1.9%, and -5.5%, respectively.

Such accuracy in predicting the known four-loop terms in the $H \rightarrow gg$ rate suggests that the prediction for c_0 in (3.12) is a credible one. One way to test the stability of this prediction is to utilize the true values of c_1, c_2, c_3 within the moment expressions (3.10) to obtain [for the numerical values (3.11) already obtained for these moments] four independent determinations of c_0 . We then find that

$$\begin{aligned} (3.10a) : \quad c_0 &= 2412, \\ (3.10b) : \quad c_0 &= 2436, \\ (3.10c) : \quad c_0 &= 2444, \\ (3.10d) : \quad c_0 &= 2447, \end{aligned} \quad (3.14)$$

results all within 2% of that in (3.12).

An alternative approach to matching the four-loop coefficients c_i within (3.4) to the error improved Padé estimate (3.8) is optimize the least-squares function [3]

$$\chi^2[c_0, c_1, c_2, c_3] = \int_0^1 [R_3 - (c_0 - c_1 \ln w + c_2 \ln^2 w - c_3 \ln^3 w)]^2 dw, \quad (3.15)$$

with R_3 in the integral given by (3.8). As before, factors of R_1 and R_2 appearing in (3.8) are given explicitly by (3.2) and (3.3) with $L = -\ln(w)$. One then finds for $M_H = 100$ GeV [$T = -1.126$] that

$$\begin{aligned} \chi^2(c_0, c_1, c_2, c_3) = & 4.064168878 \cdot 10^7 + 720c_3^2 + 4c_0c_2 + c_0^2 + 2c_0c_1 + 24c_2^2 + 12c_1c_2 + 12c_0c_3 + 2c_1^2 \\ & + 240c_2c_3 + 48c_1c_3 - 10205.87c_0 - 17507.05c_1 - 54106.23c_2 - 234536.67c_3 \end{aligned} \quad (3.16)$$

The optimization requirement

$$\frac{\partial \chi^2}{\partial c_i} = 0 \quad (3.17)$$

yields predictions remarkably close to those of (3.12),

$$c_0^{\chi^2} = 2452, \quad c_1^{\chi^2} = 1774, \quad c_2^{\chi^2} = 377.2, \quad c_3^{\chi^2} = 20.45, \quad (3.18)$$

further confirming the stability of the estimation procedure.

In Table 1, we have tabulated a set of predictions of the four-loop term c_0 for values of the Higgs mass between 100 and 175 GeV. Also tabulated are the errors in the predicted values for c_1, c_2, c_3 , relative to the true values for these coefficients, as given in (2.11-13). Estimated values of c_1 and c_2 remain within 2% of their true values (2.11-13) over the range of Higgs masses given; the relative error of c_3 estimates remains below 7% for the same range. This consistency provides further support for the c_0 estimates presented in the final column.

These c_0 estimates may be utilized to ascertain the convergence of the series (2.7). If we choose $\mu = m_t(\mu)$, the logarithmic factors L^k within (2.7) all vanish. To evaluate the series S/x^2 in (2.7), we assume that $m_t(m_t) \cong M_t = 175.6$ GeV and that $\alpha_s(175.6) = 0.10915 = \pi x(175.6)$, as evolved from $\alpha_s(M_z) = 0.119$ [12], in which case

$$S(\mu = 175.6 \text{ GeV})/x^2(175.6) = 1 + a_0x(175.6) + b_0x^2(175.6) + c_0x^3(175.6). \quad (3.19)$$

For Higgs masses between 100-175 GeV, we tabulate in Table 2 the magnitudes of successive orders in (3.19). The final 4th order term is obtained from the appropriate Padé estimate for c_0 listed in Table 1. Table 2 shows that the four-loop term decreases from 10% to 3% of the leading contribution as M_H increases from 100 to 175 GeV. Moreover, the convergence of the series is problematical in the absence of the estimate for the four loop term. Over the range of Higgs masses considered, the three-loop contribution is between 18% and 33% of the leading (one-loop) contribution.

The procedures delineated above can also be utilized to predict c_0 's explicit polynomial dependence of T ,

$$c_0(T) = a_0 + a_1T + a_2T^2 + a_3T^3, \quad (3.20)$$

analogous to the expressions (2.11-13) obtained from RG -invariance for $c_1(T)$, $c_2(T)$, and $c_3(T)$. To extract the T dependence of c_0 , we first incorporate all known T -dependence into the least-squares function (3.15):

$$\chi^2[c_0(T)] = \int_0^1 [R_3[L, T] - \{c_0(T) + c_1(T)L + c_2(T)L^2 + c_3(T)L^3\}]^2 dw. \quad (3.21)$$

In (3.21), $L \equiv -\ln w$. The quantities $c_1(T)$, $c_2(T)$ and $c_3(T)$ are no longer optimizable variables as in (3.15), but are now the explicit polynomials (2.11-13) obtained in the previous section via RG -methods. The factor $R_3[L, T]$ in the integrand of (3.21) is just (3.8) generalized to include the explicit T -dependence of R_1 and R_2 [eqs. (3.2) and (3.3)]:

$$R_3[L, T] = \frac{2R_2^3[L, T]}{R_1[L, T]R_2[L, T] + R_1^3[L, T]}. \quad (3.22)$$

From (3.21), the requirement $d\chi^2/dc_0 = 0$ generates c_0 as a function of T . Since $T \equiv \ln(M_H^2/M_t^2)$, we restrict our attention to the region $-1 \leq T \leq 0$ (*i.e.* to values for M_H between M_t and $M_t e^{-1/2} = 107$ GeV). A set of values for $c_0(T)$ can be obtained via optimization of (3.21) over values of T in this region:

$$\begin{aligned} c_0(0) &= 735.7, \quad c_0(-0.1) = 841.7, \quad c_0(-0.2) = 955.6, \quad c_0(-0.3) = 1078, \\ c_0(-0.4) &= 1208, \quad c_0(-0.5) = 1346, \quad c_0(-0.6) = 1493, \quad c_0(-0.7) = 1649, \\ c_0(-0.8) &= 1814, \quad c_0(-0.9) = 1988, \quad c_0(-1.0) = 2170. \end{aligned} \quad (3.23)$$

We obtain a least-squares fit of these results to the form (3.20) by optimizing

$$\chi^2[a_0, a_1, a_2, a_3] \equiv \sum_{i=0}^{10} \left[c_0(-i/10) - \left(a_0 - \frac{a_1 i}{10} + \frac{a_2 i^2}{100} - \frac{a_3 i^3}{1000} \right) \right]^2 \quad (3.24)$$

with respect to $\{a_0, a_1, a_2, a_3\}$, and find that

$$c_0(T) = 735.7 - 1020T + 388.8T^2 - 25.41T^3. \quad (3.25)$$

Each coefficient listed above is within 5% relative error of the corresponding coefficient obtained via a least squares fit of the c_0 values displayed in Table 1 to the polynomial form (3.20):

$$c_0(T) = 755.9 - 1029T + 394.3T^2 - 26.74T^3. \quad (3.26)$$

Note that (2.11), (2.12), (2.13) and the Padé-estimates (3.25) or (3.26) specify *all* the logarithmic coefficients within the full four loop contribution (3.4) to the $H \rightarrow gg$ decay rate (2.6,7), and comparison of these results with future perturbative calculations should yield information which can be employed to further improve Padé estimation procedures. The exact $H \rightarrow gg$ rate (2.6) is necessarily a scale invariant physical quantity. The factor $S[x(\mu), L(\mu), T]$ within (2.6) will exhibit residual scale-dependence [*i.e.* μ -dependence] only as a consequence of truncation of the series S to a given order of perturbation theory. In the section that follows, the incorporation of the four-loop coefficients c_{0-3} of $S[x(\mu), L(\mu), T]$ is seen to eliminate virtually all of this residual scale dependence.

4. The $H \rightarrow gg$ Rate

In Fig. 1, the μ -dependence of three- and four-loop expressions for $S[x^{(6)}(\mu), L(\mu), T]$ is plotted for the case of a 140 GeV Higgs mass [$T = 2 \ln(140/175.6)$]. The four-loop term within S is evaluated through use of the Padé estimate $c_0 = 1306$ (Table 1) in conjunction with eqs. (2.11-2.13) for the RG -accessible coefficients c_1, c_2 , and c_3 . The running coupling $\alpha_s(\mu)$ and running mass $m_t(\mu)$ occurring within (2.7) are evolved via four-loop β and γ -functions from physical reference values $\alpha_s(M_z) = 0.119$ and $m_t(m_t) = 175.6$ GeV [12].

Figure 1 shows that the four-loop expression eliminates virtually all of the residual scale dependence still evident in the three-loop rate in the region $M_t \geq \mu \geq 30$ GeV. The four-loop rate is observed to have a local minimum at $\mu = 43.5$ GeV and a weak local maximum at $\mu = 89.5$ GeV. The values of $S[x(\mu), L(\mu), T]$ at both of these points of minimal-sensitivity [13] differ by only 0.2%, indicative of the flatness of S between these two points. In Table 3, values for $S[x(\mu), L(\mu), T]$ as well as the term-by-term series $= x^2[1 + R_1x + R_2x^2 + R_3x^3]$ within S are displayed for a variety of μ -values of interest. The table displays the relative size of successive terms $R_n x^n$ at the minimal-sensitivity points $\mu = 43.5$ and 89.5 GeV in addition to the points $\mu = 140$ GeV ($= M_H$) and $\mu = 175$ GeV ($\cong M_t$). The relative magnitude of the four-loop term $R_3 x^3$ is seen to be less than 1% of the leading term in the series (unity) over the entire region between the minimal-sensitivity values of μ .

Of particular interest in this range are those values of μ [47.0 GeV and 73.5 GeV] at which the four-loop term $R_3 x^3$ effectively vanishes. These values correspond to the two points in Figure 1 at which the 3-loop and 4-loop curves cross. The usual approach towards extracting information from an asymptotic series $\sum_{n=0} R_n x^n$ ($R_0 = 1$) is to sum only that series' decreasing terms, *i.e.* to evaluate $\sum_{n=0}^{n'} R_n x^n$ by choosing n' such that $|R_{n'} x^{n'}|$ is a minimum. By choosing μ so as to have $R_3(\mu)$ *vanish*, one can then argue (for this choice of μ) that $n' = 3$, suggesting that such a value for μ is optimal for estimating the series from its *known* terms. Of course, such an

interpretation rests on the assumed increase of terms $|R_n x^n|$ subsequent to $n' = 3$; all we can really be certain of is that $|R_4 x^4| > |R_3 x^3|$ ($= 0$) at such a value of μ . Nevertheless, Table 3 shows that both values of μ for which $R_3 \rightarrow 0$ lie between the two minimal-sensitivity points [$\mu = 43.5$ and 89.5 GeV], and that the μ -sensitive factor $S[x(\mu), L(\mu), T]$ within the rate (2.6) varies by less than 0.2% over this entire region. Over the full range of μ values displayed in Table 3 ($43.5 \text{ GeV} < \mu < 175 \text{ GeV}$), the four-loop series term ($R_3 x^3$) varies between 0 and 5% of the leading one-loop order term (unity). Surprisingly, however, the rate $S[x(\mu), L(\mu), T]$ displays a relative spread of values $\Delta S/S \leq 0.4\%$ over the same range of μ , indicative of a substantial reduction in residual scale dependence.

In Figures 2 and 3 we exhibit the residual scale dependence of $S[x(\mu), L(\mu), T]$ to three and four-loop order for Higgs masses of 100 GeV and 175 GeV, respectively. These figures show the same reduction in scale dependence evident in Figure 1 when $M_H = 140$ GeV. For the case of $M_H = 100$ GeV, for example, Table 4 shows that the 4-loop term $R_3 x^3$ varies between zero and 10% of the leading-order 1-loop term between the local minimum at $\mu = 29$ GeV and $\mu \cong M_t$. Over this same range of μ , the four-loop values of $S[x(\mu), L(\mu), T]$ remain within 1% of each other. For $M_H = 175$ GeV, the four-loop term is between zero and 3% of the leading-order term between the local minimum at $\mu = 48.5$ GeV and $\mu \cong M_t$, whereas the full four-loop expression for $S[x(\mu), L(\mu), T]$ exhibits a relative spread of only 0.1% [Table 5].

Such scale independence is, of course, a reflection of the RG -invariance (2.8) of $S[x(\mu), L(\mu), T]$, which has been utilized explicitly to obtain the coefficients c_{1-3} . Nevertheless, the coefficient c_0 , which is *not* perturbatively accessible via (2.8) but is obtainable (at present) only by Padé approximant methods, appears to be precisely what is required to eliminate virtually all residual μ -dependence in the rate arising from truncation. For example, if $M_H = 100$ GeV, the factor $c_0 x^3(\mu)$ in isolation contributes more than 10% of the leading order contribution (unity) for $\mu \lesssim M_t$. Moreover, this contribution *increases* as μ decreases. Nevertheless, the particular choice $c_0 = 2453$ appears to ensure that the overall spread in S remains within 1%, despite the potentially large contribution to this spread from $c_0 x^3(\mu)$. Evidently the μ dependence of this term serves to cancel the residual μ -dependence of the remaining terms in the series.¹

This near cancellation of residual scale-dependence makes possible a set of credible predictions for the rate (2.6). In the fourth column of Table 6 we have tabulated the $H \rightarrow gg$ decay rates for $M_H = 100, 125, 140, 150$, and 175 GeV in the $m_b \rightarrow 0$, $M_H^2 \ll 4M_t^2$ limit. The largest source of uncertainty for these predictions is in the value for $\alpha_s(M_z) = (0.119 \pm 0.002$ [12]), which should lead to 4% uncertainty in the rates presented in Table 6.

Fermion mass effects (*i.e.* the departures from the $m_b \rightarrow 0$, $M_H^2 \ll 4M_t^2$ assumptions implicit in the derivation [1] of the $H \rightarrow gg$ rate) can be accommodated in leading order by replacing the factor of unity in (3.1) with the following m_b - and M_t -sensitive terms [14]:

$$1 \rightarrow 1 + \delta_m = \frac{9}{16} [(A_t + \text{Re}A_b)^2 + (\text{Im}A_b)^2], \quad (4.1)$$

$$A_t = 2 \left[\tau_t + (\tau_t - 1) (\sin^{-1}(\sqrt{\tau_t}))^2 \right] / \tau_t^2, \quad \tau_t \equiv M_H^2 / 4M_t^2, \quad (4.2)$$

$$A_b = 2 [\tau_b + (\tau_b - 1)f(\tau_b)] / \tau_b^2, \quad \tau_b \equiv M_H^2 / 4m_b^2, \quad (4.3)$$

$$f(\tau) \equiv -\frac{1}{4} \left[\ln \left(\frac{1 + \sqrt{1 - 1/\tau}}{1 - \sqrt{1 - 1/\tau}} \right) - i\pi \right]^2. \quad (4.4)$$

The right hand side of (4.1) is easily seen to approach unity when $m_b \rightarrow 0$, $M_H^2 / 4M_t^2 \rightarrow 0$:

$$\lim_{\tau_t \rightarrow 0} A_t = \frac{4}{3}; \quad \lim_{\tau_b \rightarrow \infty} A_b = 0. \quad (4.5)$$

The leading mass correction δ_m , as defined in (4.1), is tabulated for various Higgs boson masses (with PDG [12] fermion-mass values $m_b = 4.2$ GeV, $M_t = 175.6$ GeV) in the fifth column of Table 6. This lowest-order fermion mass correction can be incorporated into the rate (2.6) by its inclusion into the series (3.1)

$$S = x^2 [1 + R_1 x + R_2 x^2 + R_3 x^3 + \delta_m]. \quad (4.6)$$

¹The reduction of scale-dependence via Padé approximant methods is discussed in detail by Gardi [5].

The sixth column of Table 6 tabulates the $H \rightarrow gg$ rate with this correction included.

Because δ_m is a leading order correction, there is genuine ambiguity as to whether "physical" or running masses should be incorporated into this correction. The former choice yields a manifestly scale-dependent contribution $\Delta S = \delta_m^p x^2(\mu)$ to the rate S . One could argue for the incorporation of *running masses* $m_b(\mu), m_t(\mu)$ in (4.2) and (4.3); *i.e.*, $\Delta S = \delta_m(\mu) x^2(\mu)$, with $\delta_m(\mu)$ calculated via (4.1-4) with $\tau_t = M_H^2/4m_t^2(\mu)$ and $\tau_b = M_H^2/4m_b^2(\mu)$, and with μ identified consistently with the minimal-sensitivity scale tabulated in Column 2 of Table 6. In Column 7 of Table 6 we have tabulated $\delta_m(\mu)$ utilizing 6-active-flavour running masses referenced to $m_t(m_t) = 175.6$ GeV, $m_b(m_b) = 4.2$ GeV, and in Column 8 we have listed the corresponding $H \rightarrow gg$ decay rates. Theoretical uncertainty associated with the masses utilized in the leading correction-factor δ_m is reflected in the differing rates of Column 6 and Column 8. This discrepancy is seen to be at most a 2% effect.

The leading order mass-correction factor δ_m in (4.6) is seen from a comparison of Table 6 to Tables 3-5 to be generally smaller than the three-loop term $R_2 x^2$ in (4.6), but somewhat larger than the four-loop term $R_3 x^3$ also appearing in (4.6) [δ_m is smaller than $R_3 x^3$ when $\mu = M_t$, as in Table 2]. Since the next order of fermion mass corrections is suppressed by an additional power of α_s , we anticipate such next-to-leading-order fermion mass corrections to be well within 1% of the total rate. In any case, both leading-order mass corrections δ_m and the four-loop contribution $R_3 x^3$ we obtain are sufficiently small for credible estimates of the $H \rightarrow gg$ rate on the basis of the terms in (4.6). It is also worth noting from Table 6 that δ_m^p itself is only 1% of the leading-order one loop term [normalized to unity in (4.6)] when $M_H = 140 - 150$ GeV, and that $\delta_m(\mu)$ is comparably small when $M_H = 150$ GeV. Hence, the rates tabulated in Table 6 are optimally valid for Higgs masses in the 140-150 GeV range.

5. The $H \rightarrow b\bar{b}$ rate

The leading QCD scalar-current correlation function contributions to the *Higgs* $\rightarrow b\bar{b}$ decay rate are known from explicit calculation to $\mathcal{O}(\alpha_s^3)$, with secondary $\mathcal{O}(m_b^2/M_H^2)$ power corrections known to $\mathcal{O}(\alpha_s^2)$, as given in [7]:

$$\begin{aligned} \Gamma(H \rightarrow b\bar{b}) = & \frac{3G_F}{4\sqrt{2}\pi} M_H m_b^2(M_H) \left[\Pi_{scalar}(\mu^2 = s = M_H^2) \right. \\ & \left. + \frac{m_b^2(M_H)}{M_H^2} \left(-6 - 40 \frac{\alpha_s(M_H)}{\pi} - 87.72 \left(\frac{\alpha_s(M_H)}{\pi} \right)^2 \right) \right]. \end{aligned} \quad (5.1)$$

The factor Π_{scalar} in (5.1) is the imaginary part of the QCD correlation function for the quark-antiquark scalar current normalized to unity in leading order [7]:

$$\begin{aligned} \Pi_{scalar}[\mu, s, x(\mu)] = 1 & + \left(\frac{17}{3} + 2 \ln \mu^2/s \right) x(\mu) \\ & + \left(29.1467 + \frac{263}{9} \ln \left(\frac{\mu^2}{s} \right) + \frac{47}{12} \ln^2 \left(\frac{\mu^2}{s} \right) \right) x^2(\mu) \\ & + \left(41.7576 + 238.381 \ln \left(\frac{\mu^2}{s} \right) + 94.6759 \ln^2 \left(\frac{\mu^2}{s} \right) + 7.61574 \ln^3 \left(\frac{\mu^2}{s} \right) \right) x^3(\mu) \\ & + \left(d_0 + d_1 \ln \left(\frac{\mu^2}{s} \right) + d_2 \ln^2 \left(\frac{\mu^2}{s} \right) + d_3 \ln^3 \left(\frac{\mu^2}{s} \right) + d_4 \ln^4 \left(\frac{\mu^2}{s} \right) \right) x^4(\mu) \\ & + \dots \end{aligned} \quad (5.2)$$

where $x(\mu) = [\alpha_s(\mu)]_{n_f=5}/\pi$, corresponding to five active flavours.

In ref.[2], a detailed asymptotic Padé-approximant procedure is presented for estimating the coefficients d_{0-4} in (5.2). The methodology is virtually identical to that of Section 3 above, except that five moments are now calculated for $R_4[w]$, the coefficient of $x^4(\mu)$, based upon the asymptotic error formula prediction [11]

$$R_4 = \frac{R_3^2[R_2^3 + R_1 R_2 R_3 - 2R_1^3 R_3]}{R_2[2R_2^3 - R_1^3 R_3 - R_1^2 R_2^2]}. \quad (5.3)$$

The results, as tabulated in Table 3 of ref. [2], are

$$d_0 = 64.2, \quad d_1 = 745, \quad d_2 = 1180, \quad d_3 = 253, \quad d_4 = 15.4. \quad (5.4)$$

The factors d_{1-4} may be obtained directly from RG-invariance of the physical rate

$$\mu^2 \frac{d}{d\mu^2} [m_b^2(\mu) \Pi_{\text{scalar}} [\mu, s, x(\mu)]] = 0, \quad (5.5)$$

leading to the following values (also tabulated) in [2]):

$$d_1 = 791.52, \quad d_2 = 1114.7, \quad d_3 = 260.06, \quad d_4 = 14.755. \quad (5.6)$$

The strong agreement between (5.6) and (5.4) suggests that the estimate for d_0 in (5.4) is a credible one. It is interesting to note that a much more naive Padé approach for estimating d_0 , in which (5.3) was applied directly to $\Pi_{\text{scalar}}(\mu^2 = s = M_H^2)$, yielded a value for d_0 ($=67.25$ [11]) surprisingly consistent with the estimate in (5.4). Only the latter estimate exhibits sensitivity to the logarithmic terms in Π_{scalar} , which all vanish when $\mu^2 = s$.

If we utilize the estimate for d_0 quoted in (5.4) within the $H \rightarrow b\bar{b}$ rate, we find that the correlation-function factor within (5.1) is given to $\mathcal{O}(\alpha_s^4)$ by

$$\Pi_{\text{scalar}}(\mu^2 = s = M_H^2) = 1 + \frac{17}{3}x(M_H) + 29.1467 x^2(M_H) + 41.7576 x^3(M_H) + \underline{64} x^4(M_H). \quad (5.7)$$

The underlined coefficient is, of course, d_0 as estimated in [2] via asymptotic Padé-approximant methods.

There is genuine value in having an estimate of this term, as it is generally larger than the $\mathcal{O}(\alpha_s^2 m_b^2/M_H^2)$ final term in (5.1). For example, suppose that $M_H = 130$ GeV, $m_b(130 \text{ GeV}) = 2.7$ GeV and $x(130 \text{ GeV}) = 0.114/\pi$ (these values are consistent with those employed in Table 1 of ref. [7]). The relative magnitudes of the correlator contributions 95.5) to the $H \rightarrow b\bar{b}$ rate are seen to be

$$\Pi_{\text{scalar}}(\mu^2 = s = (130 \text{ GeV})^2) = 1 + 0.2056 + 0.0384 + 0.0020 + \underline{0.00011}. \quad (5.8)$$

All but the underlined Padé-estimate term are tabulated in Table 1 of ref. [7]. Comparison to the power-suppressed terms in (5.1),

$$\frac{m_b^2(M_H)}{M_H^2} (-6 - 40 x(M_H) - 87.72 x^2(M_H)) \xrightarrow{M_H=130 \text{ GeV}} -0.0026 - 0.00062 - 0.00005, \quad (5.9)$$

(also tabulated in [7]) reveals that the $\mathcal{O}(\alpha_s^4)$ term in (5.8) is double the magnitude of the $\mathcal{O}(\alpha_s^2)$ term in (5.9). Hence the Padé-estimated $\mathcal{O}(\alpha_s^4)$ term in (5.7) enables one to utilize the full precision available in the known power-suppressed contributions to (5.1). However, one cannot anticipate such precision experimentally for many years to come.

6. Summary

In the preceding sections, we have demonstrated how asymptotic Padé-approximant methods may be utilized to estimate the unknown $\mathcal{O}(\alpha_s^5)$ coefficients $\{c_0, c_1, c_2, c_3\}$ within the $H \rightarrow 2g$ decay rate (2.7). Such estimates for $\{c_1, c_2, c_3\}$ are seen (Table 1) to be within a few percent of the true values for these coefficients, which can be extracted via renormalization-group methods. This accuracy supports corresponding asymptotic Padé-approximant estimates of the renormalization-group-inaccessible coefficients c_0 presented in Table 1. Moreover, the inclusion of $\mathcal{O}(\alpha_s^5)$ terms within (2.7) is seen to virtually eliminate the scale-parameter dependence of the rate over an astonishingly large range of the scale-parameter μ , typically $0.3M_H \lesssim \mu \lesssim M_t$. Inclusion of estimated $\mathcal{O}(\alpha_s^5)$ corrections in conjunction with leading-order fermion-mass corrections to the rate (Table 6) are seen to reduce the perturbative uncertainty in the $H \rightarrow 2g$ decay rate from $\mathcal{O}(20\%)$ to $\mathcal{O}(2\%)$. Four-loop corrections to the $H \rightarrow b\bar{b}$ decay mode are also presented, which are seen to reduce the very small perturbative uncertainty of this dominant hadronic mode by an additional order of magnitude.

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References

- [1] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, Phys. Rev. Lett. **79** (1997) 353.
- [2] F. Chishtie, V. Elias, and T.G. Steele, Phys. Rev. D **59** (1999) 105013.
- [3] F. A. Chishtie, V. Elias, and T.G. Steele, J. Phys. G **26** (2000) 93.
- [4] J. Ellis, E. Gardi, M. Karliner and M. A. Samuel, Phys. Lett. B **366** (1996) 268 and Phys. Rev. D **54** (1996).
- [5] E. Gardi, Phys. Rev. D **56** (1997) 68.
- [6] M.R. Ahmady, F. A. Chishtie, V. Elias, and T.G. Steele, Phys. Lett. B (to appear: hep-ph/9910551).
- [7] K. G. Chetyrkin, Phys. Lett. B **390** (1997) 309.
- [8] T. van Ritbergen, Phys. Lett. B **454** (1999) 353.
- [9] S. A. Larin, T. van Ritbergen, and J. A. M. Vermaseren, Nucl. Phys. B **438** (1995) 278.
- [10] J. Ellis, I. Jack, D.R.T. Jones, M. Karliner, and M. Samuel, Phys. Rev D **57** (1998) 2665.
J. Ellis, M. Karliner, and M. A. Samuel, Phys. Lett. B **400** (1997) 176.
- [11] V. Elias, T.G. Steele, F. Chishtie, R. Migneron, and K. Sprague, Phys. Rev. D **58** (1998) 116007.
- [12] C. Caso et. al. [Particle Data Group], Eur. Phys. J. C **3** (1998) 1.
- [13] P.M. Stevenson, Phys. Rev. D **23** (1981) 2916.
- [14] M. Spira, A. Djouadi, D. Graudenz and P.M. Zerwas, Nucl. Phys. B **453** (1995) 17.

$M_H(\text{GeV})$	$\delta c_1/c_1^{RG}$	$\delta c_2/c_2^{RG}$	$\delta c_3/c_3^{RG}$	c_0
100	-0.010	-0.019	-0.054	2453
125	-0.009	-0.013	-0.059	1646
140	-0.009	-0.009	-0.062	1306
150	-0.009	-0.007	-0.068	1120
175	-0.012	-0.0004	-0.070	763

Table 1: The final column tabulates asymptotic Padé-approximant predictions for the four loop nonlogarithmic coefficient c_0 within (2.7), as obtained via moments of $c_0 - c_1 \ln w + c_2 \ln^2 w - c_3 \ln^3 w$, for Higgs masses between 100 GeV and M_t . Relative errors between predictions and true values of c_1, c_2 and c_3 , as determined from RG invariance, are tabulated in the second, third and fourth column: $\delta c_i = (c_i^{\text{predicted}} - c_i^{RG})/c_i^{RG}$.

$M_H(\text{GeV})$	$S[x(M_t), L(M_t), T]/x^2 = 1 + a_0 x + b_0 x^2 + c_0 x^3$
100	$1 + 0.7728 + 0.3333 + \underline{0.1014}$
125	$1 + 0.7133 + 0.2675 + \underline{0.0679}$
140	$1 + 0.6831 + 0.2361 + \underline{0.0538}$
150	$1 + 0.6647 + 0.2177 + \underline{0.0461}$
175	$1 + 0.6237 + 0.1783 + \underline{0.0312}$

Table 2: The perturbative series S/x^2 within the $H \rightarrow 2g$ decay rate (2.7), evaluated at $\mu = m_t(\mu) = 175.6 \text{ GeV}$. $x(175.6 \text{ GeV}) = 0.10915/\pi$, consistent with $x(M_z) = \alpha_s(M_z)/\pi = 0.119/\pi$. The final underlined term of the series is obtained from the c_0 estimates in Table 1.

$\mu(\text{GeV})$	$\alpha_s(\mu)$	$m_t(\mu)(\text{GeV})$	$1 + R_1 x + R_2 x^2 + R_3 x^3$	$S[x(\mu), L(\mu), T]$
43.5	0.1325	198.2	$1 + 0.3811 - 0.0395 + 0.0023$	0.002390
47.0	0.1309	196.8	$1 + 0.4014 - 0.0251 - 0.0001$	0.002390
73.5	0.1226	188.8	$1 + 0.5094 + 0.0614 + 0.0000$	0.002393
89.5	0.1193	185.6	$1 + 0.5525 + 0.1007 + 0.0068$	0.002394
140	0.1124	178.8	$1 + 0.6419 + 0.1907 + 0.0344$	0.002390
175	0.1092	175.6	$1 + 0.6825 + 0.2354 + 0.0534$	0.002384

Table 3: The four-loop expression for the scale-dependent factor $S[x(\mu), L(\mu), T]$ for $M_H = 140 \text{ GeV}$ within the Higgs \rightarrow gluon-gluon decay rate (2.6) for five different values of μ . Weak extrema within the rate (Fig. 1) occur at $\mu = 43.5$ and 89.5 GeV . Also displayed are $\mu = M_H$, $\mu \approx M_t$, and two values of μ (47.0 GeV and 73.5 GeV) at which the four-loop contribution to S is almost zero. The values of S over this entire range of μ are seen to be remarkably static, as discussed in the text.

$\mu(\text{GeV})$	$\alpha_s(\mu)$	$m_t(\mu) (\text{GeV})$	$1 + R_1 x + R_2 x^2 + R_3 x^3$	$S[x(\mu), L(\mu), T]$
29.0	0.1413	206.5	$1 + 0.3819 - 0.0546 + 0.0033$	0.002691
31.0	0.1397	205.1	$1 + 0.4007 - 0.0410 + 0.0002$	0.002691
54.0	0.1282	194.2	$1 + 0.5418 + 0.0775 + 0.0001$	0.002698
65.5	0.1247	190.8	$1 + 0.5853 + 0.1198 + 0.0089$	0.002699
100	0.1175	183.8	$1 + 0.6722 + 0.2125 + 0.0402$	0.002693
175	0.1092	175.6	$1 + 0.7722 + 0.3326 + 0.1010$	0.002667

Table 4: The scale-dependence of the four-loop expression for $S[x(\mu), L(\mu), T]$ is displayed, as in Table 3, for the case $M_H = 100 \text{ GeV}$. Weak extrema in Figure 2 occur at $\mu = 29.0$ and 65.5 GeV . The four-loop contribution is seen to nearly vanish at $\mu = 31.0$ and 54.0 GeV .

$\mu(\text{GeV})$	$\alpha_s(\mu)$	$m_t(\mu) (\text{GeV})$	$1 + R_1x + R_2x^2 + R_3x^3$	$S[x(\mu), L(\mu), T]$
48.5	0.1303	196.2	$1 + 0.3385 - 0.0584 + 0.0081$	0.002216
61.0	0.1260	192.0	$1 + 0.3976 - 0.0190 - 0.0001$	0.002216
91.0	0.1190	185.3	$1 + 0.4912 + 0.0537 + 0.0000$	0.002218
111	0.1159	182.2	$1 + 0.5338 + 0.0910 + 0.0058$	0.002218
175	0.1092	175.6	$1 + 0.6230 + 0.1777 + 0.0309$	0.002215

Table 5: The scale-dependence of the four-loop expression for $S[x(\mu), L(\mu), T]$ is displayed, as in Table 3, for the case $M_H = 175$ GeV. Weak extrema in Figure 3 occur at $\mu = 48.5$ and 111.0 GeV. The four-loop term is seen to nearly vanish at $\mu = 61.0$ and 91.0 GeV. The relative spread in the rate S is seen to be only of order 0.1% for values of μ between 48.5 and 175 GeV.

M_H	μ	S	$\Gamma_{H \rightarrow gg}^{\delta_m=0}$	δ_m^p	$\Gamma_{H \rightarrow gg}^{\delta_m^p}$	$\delta_m(\mu)$	$\Gamma_{H \rightarrow gg}^{\delta_m(\mu)}$
100	65.5	0.00270	$1.97 \cdot 10^{-4}$	-0.085	$1.87 \cdot 10^{-4}$	-0.059	$1.90 \cdot 10^{-4}$
125	80.5	0.00249	$3.55 \cdot 10^{-4}$	-0.040	$3.47 \cdot 10^{-4}$	-0.015	$3.52 \cdot 10^{-4}$
140	89.5	0.00239	$4.79 \cdot 10^{-4}$	-0.012	$4.76 \cdot 10^{-4}$	+0.011	$4.82 \cdot 10^{-4}$
150	95.5	0.00234	$5.75 \cdot 10^{-4}$	+0.0076	$5.78 \cdot 10^{-4}$	+0.023	$5.83 \cdot 10^{-4}$
175	111	0.00222	$8.67 \cdot 10^{-4}$	+0.060	$8.99 \cdot 10^{-4}$	+0.077	$9.08 \cdot 10^{-4}$

Table 6: The four-loop order $H \rightarrow gg$ decay rate (2.6) is tabulated for various choices of M_H . The second column lists the minimal-sensitivity values of μ ($dS/d\mu = 0$) closest to M_H . The third column is the value of $S[x(\mu), L(\mu), T]$ evaluated at this choice for μ , as indicated in the previous three tables. The fourth column is the rate in the $m_b \rightarrow 0$, $M_H^2 \ll 4M_t^2$ limit. The fifth column tabulates the magnitudes of the leading fermion mass correction to S/x^2 , as discussed in the text, using "physical" (p) fermion masses $M_t = 175.6$ GeV, $m_b = 4.2$ GeV. The sixth column tabulates the $H \rightarrow gg$ rate incorporating this mass correction. The seventh column lists the leading fermion mass correction to S/x^2 utilizing running fermion mass values $m_b(\mu), m_t(\mu)$ at the Column 2 values of μ , and the final column tabulates the $H \rightarrow gg$ rate incorporating this fermion-mass correction. All masses and decay rates are in GeV units.

Fig.1

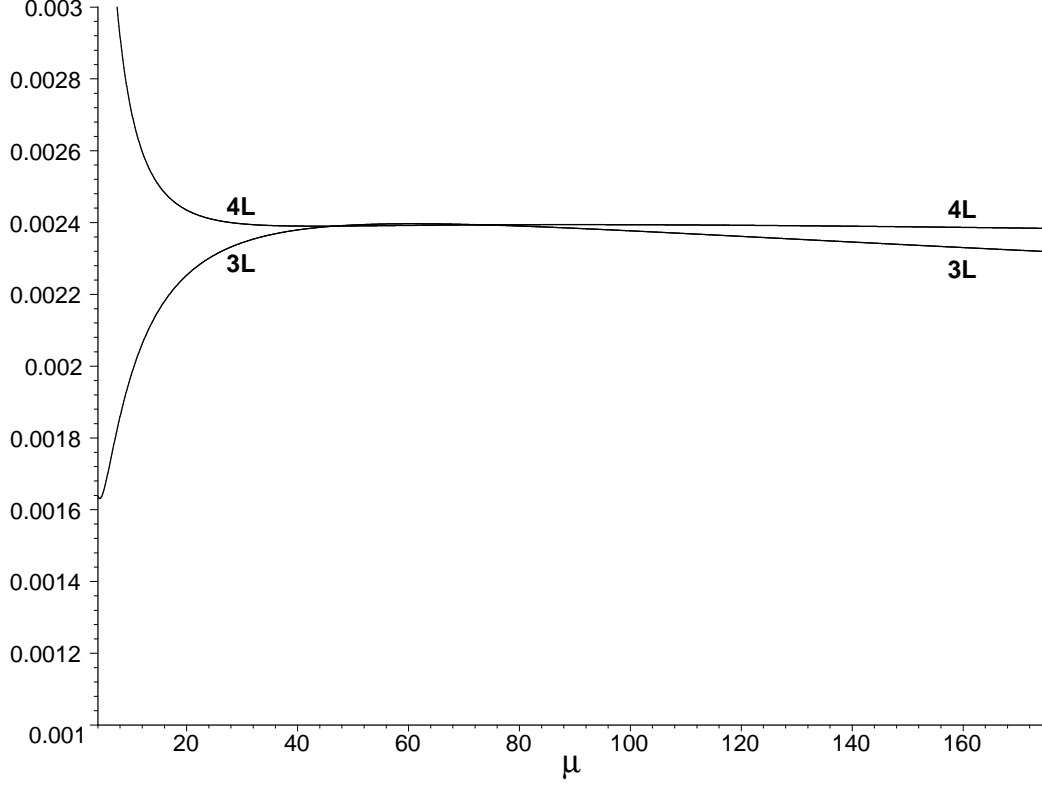


Figure 1: Comparison of the residual renormalization scale- (μ -) dependence of the three-loop (3L) and four-loop (4L) expression for the $H \rightarrow gg$ decay rate of a Higgs boson with mass $M_H = 140$ GeV. The y -axis numbers correspond to the dimensionless scale-dependent factor $S[x(\mu), L(\mu), T]$ within the rate (2.6), and the x -axis is the scale μ in GeV. The logarithmic coefficients of 4L-contributions are extracted via renormalization-group equation methods. The 4L non-logarithmic coefficient is obtained via asymptotic Padé approximant methods. The 3L and 4L curves nearly coincide in the region $44 \text{ GeV} \lesssim \mu \lesssim 80 \text{ GeV}$, indicating a very small 4L contribution over this range of μ .

Fig.2

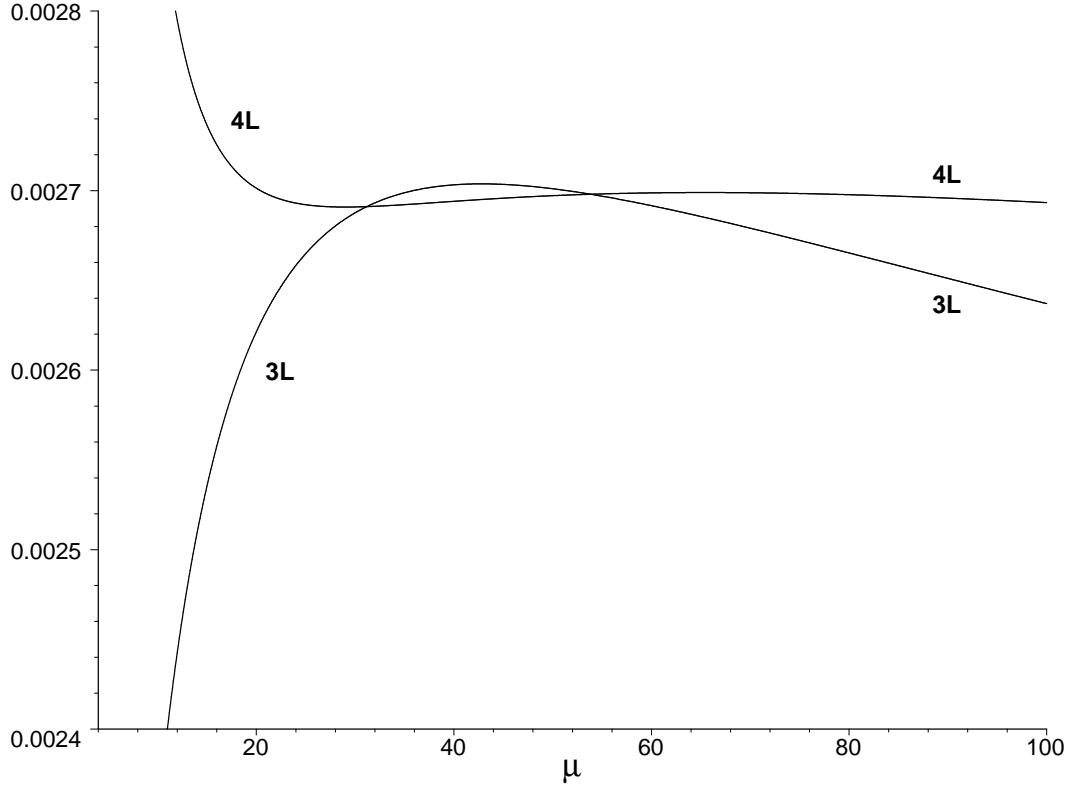


Figure 2: A comparison of the μ -dependence of three-loop and four-loop expressions for the $H \rightarrow gg$ rate is plotted, as in Figure 1, but now for $M_H = 100$ GeV. The y -axis numbers correspond to the dimensionless scale-dependent factor $S[x(\mu), L(\mu), T]$ within the rate (2.6), and the x -axis is the scale μ in GeV. The curves cross at values of μ for which the (estimated) four-loop contribution to the rate is zero.

Fig.3

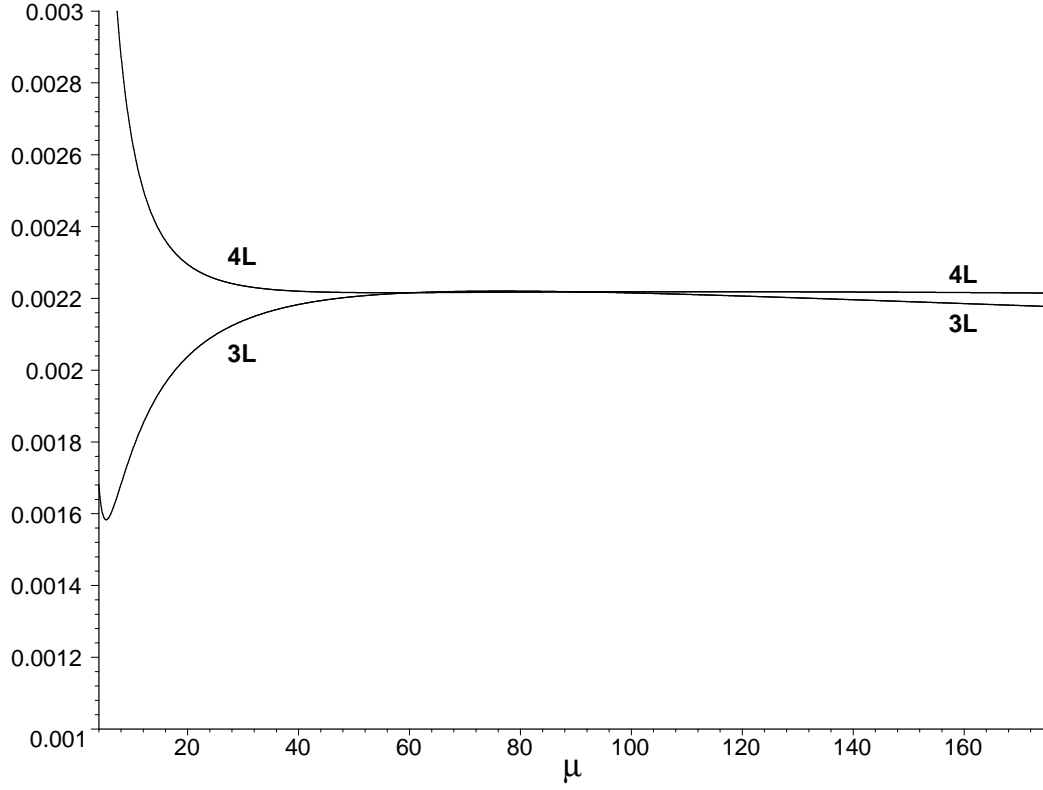


Figure 3: A comparison of the μ -dependence of three-loop and four-loop expressions for the $H \rightarrow gg$ rate (as in Figs. 1 and 2), for $M_H = 175$ GeV. The y -axis numbers correspond to the dimensionless scale-dependent factor $S[x(\mu), L(\mu), T]$ within the rate (2.6), and the x -axis is the scale μ in GeV.